Name: ____________________ Section: ____________________

Define: A set V is a *vector space* if you can *scale* and *add* vectors in V, and

1. V is closed under scaling
2. V is closed under adding
3. addition and scaling works as usual.

Define: Let V be a vector space. A subset $H \subseteq V$ is a subspace

1. $\vec{\mathbf{0}} \in H$
2. H is closed under adding
3. H is closed under scaling

Theorem:

$H \subseteq V$ is a subspace of V

and only if

there is a set of vectors $\mathscr{U}$ so that $H = Span\ \mathscr{U}$

Name: ______________________ Section: ______________________

1. Let $\mathbb{P}_2 = \{ \quad at^2 + bt + c \quad : \quad a, b, c \in \mathbb{R} \quad \}$.

   Show that $\mathbb{P}_2$ satisfies the first *three* properties of a real vector space, by showing that $\mathbb{P}_2$ is closed under scaling and adding, and that addition is commutative.

1. closed under adding: if $\vec{u}, \vec{v} \in \mathbb{P}_2$, then $(\vec{u} + \vec{v}) \in \mathbb{P}_2$

let a,b,c and d,e,f $\in \mathbb{R}$ be arbitrary

$$\vec{u} + \vec{v} = (at^2 + bt + c) + (dt^2 + et + f)$$
$$= (a+d)t^2 + (b+e)t + (c+f)$$

NOTE $\vec{u} + \vec{v} \in \mathbb{P}_2$, as desired.

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2. Closed under scaling: if $\vec{u} \in \mathbb{P}_2$ and $r \in \mathbb{R}$ then $(r \cdot \vec{u}) \in \mathbb{P}_2$

let a,b,c and r $\in \mathbb{R}$ be arbitrary.

$$r \cdot \vec{u} = r \cdot (at^2 + bt + c)$$
$$= (ra)t^2 + (rb)t + (rc) \in \mathbb{P}_2 \quad \checkmark$$

3a, adding is commutative if $\vec{u}, \vec{v} \in \mathbb{P}_2$, then $\vec{u} + \vec{v} = \vec{v} + \vec{u}$

let a,b,c, d,e,f $\in \mathbb{R}$ be arbitrary

$$\vec{u} + \vec{v} = (at^2 + bt + c) + (dt^2 + et + f)$$
$$= (a+d)t^2 + (b+e)t + (c+f)$$
$$= (d+a)t^2 + (e+b)t + (f+c) \quad \text{because adding \#'s is commutative}$$
$$= (dt^2 + et + f) + (at^2 + bt + c) = \vec{v} + \vec{u} \quad \checkmark$$

Note: you can use similar arguments to show that $\mathbb{P}_2$ *also* satisfies the other *seven* properties, showing that $\mathbb{P}_2$ is indeed a vector space.

Name: ____________________ Section: ____________________

2. Let

$$H = \left\{ \begin{bmatrix} 2p - 3q \\ 4p \\ 1 + q \end{bmatrix} \in \mathbb{R}^3 : \ p, q \text{ are in } \mathbb{R} \right\}$$

Determine if H is a subspace of $\mathbb{R}^3$. If it is, write it as the span of a set of vectors.

<u>Claim:</u> H is <u>NOT</u> a subspace.

we will prove that part ① of the defn. fails (that $\vec{0} \notin H$).

<u>proof</u> (by computation)

$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in H \iff$ there are $p, q \in \mathbb{R}$ s.t.

$$\begin{aligned} 2p - 3q &= 0 \\ 4p &= 0 \\ 1 + q &= 0 \end{aligned}$$

So we must have

$p = 0$

and $q = -1$

But then $2p - 3q = 3 \neq 0$.

<u>So</u> $\vec{0} \in H$

<u>So</u> H is <u>NOT</u> a subspace.

Name: ______________________ Section: ______________________

3. Let

$$H = \left\{ \begin{bmatrix} 3s + 2t \\ t - 3s \\ 2t + s \end{bmatrix} \in \mathbb{R}^3 : \ s, t \text{ are in } \mathbb{R} \right\}$$

Determine if H is a subspace of $\mathbb{R}^3$. If it is, write it as the span of a set of vectors.

<u>Claim</u>: H <u>is</u> a subspace.

we will prove this by rewriting H as a span.

$$H = \left\{ \begin{bmatrix} 3s + 2t \\ -3s + t \\ s + 2t \end{bmatrix} \in \mathbb{R}^3 : s, t \in \mathbb{R} \right\}$$

$$= \left\{ s\begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix} + t\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} : s, t \in \mathbb{R} \right\}$$

$$= \left\{ \vec{w} \in \mathbb{R}^3 : \vec{w} = s\begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix} + t\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \text{ for } s, t \in \mathbb{R} \right\}$$

$$= \text{Span}\left\{ \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \right\}$$

therefore H is a Span.
because every Span is a <u>space</u>,
it follows that H is a <u>space.</u>